A 51—I

OXFORD LOCAL EXAMINATIONS

General Certificate of Education

Summer Examination, 1964

Advanced Level

PURE MATHEMATICS, PAPER I

Thursday, 18 June. Time allowed: 3 hours

Write the number of the paper, **A 51/I,** *on the left at the head of each sheet of your answers in the space provided.*

Complete answers to a few questions receive more credit than scrappy attempts at the whole paper. A pass may be obtained on good answers to about **four** *questions.*

Mathematical tables and squared paper are provided.

1. Prove that, if a, b, c are the roots of the cubic equation $t^3-3dt-p=0$, then

$$(x^2+ax+d)(x^2+bx+d)(x^2+cx+d) \equiv x^6+px^3+d^3.$$

Express x^6-20x^3+343 as the product of three quadratic factors.

2. If p and q are any real numbers such that $p^2 > q^2$, prove that $p-q$ and $p+q$ are either both positive or both negative.

Find the condition that the function

$$\frac{ax^2+bx+c}{cx^2+bx+a},$$

where a, b, c and x are real and $a \neq c$, can take, for suitably chosen x, the real value y. Show that (i) if $4ac < b^2 < (a+c)^2$ there are two values between which y must not lie and that (ii) if $b^2 < 4ac$ there are two values at or between which y must lie.

3. Show that the number of ways in which n letters may be arranged in order in a row, if r of the letters are the same, is $n!/r!$.

If three boat clubs have respectively p, q and r boats, in how many ways can the boats be arranged in order, subject to the restriction that the first boat of any club is to be above its second, its second above its third, and so on?

4. Express $\dfrac{1}{(2r-1)2r(2r+1)(2r+2)}$ in partial fractions.

Show that the sum to n terms of the series whose rth term is

$$\frac{1}{(2r-1)2r(2r+1)(2r+2)}$$

is $$-\frac{5}{12}+\frac{1}{2(2n+1)}-\frac{1}{6(2n+2)}+\frac{2}{3}\left(1-\frac{1}{2}+\frac{1}{3}-\dots-\frac{1}{2n}\right)$$

and that its sum to infinity is $\frac{2}{3}\log_e 2-\frac{5}{12}$.

5. By considering the coefficient of x^p in the identity

$$(1-x)^{p+r}(1+x+x^2+\dots) \equiv (1-x)^{p+r-1},$$

where p and r are integers greater than 0, prove that

$$\frac{1}{(p+r)!}-\frac{1}{(p+r-1)!1!}+\frac{1}{(p+r-2)!2!}-\dots+(-1)^p\frac{1}{p!r!}$$
$$=\frac{(-1)^p}{(p+r)p!(r-1)!}.$$

Show that the coefficient of x^s in the expansion of

$$\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\dots+(-1)^p\frac{x^p}{p!}\right)e^x$$

in ascending powers of x is 0 if $1 \leqslant s \leqslant p$, and is

$$\frac{(-1)^p}{(p+r)p!(r-1)!} \quad \text{if} \quad s=p+r.$$

6. (*a*) Prove that the equation $(1-x^2)\dfrac{dy}{dx}-xy+1=0$ is satisfied by $y=\dfrac{\cos^{-1}x}{\sqrt{(1-x^2)}}$ and by $y=\dfrac{\log\{x+\sqrt{(x^2-1)}\}}{\sqrt{(x^2-1)}}$.

(*b*) Prove, by induction or otherwise, that, if $y=\cot^{-1}x$,

$$\frac{d^ny}{dx^n}=(-1)^n(n-1)!\sin ny \sin^n y.$$

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7. If $y = 4x \log x - x^2 - 2x + 3$, sketch the graphs, in that order, of $\dfrac{d^2y}{dx^2}$, $\dfrac{dy}{dx}$ and y for values of x greater than 0 and show that y has a maximum, a minimum and a point of inflexion. Give your reasons for the essential features of the graphs you draw.

8. (*a*) Integrate with respect to x

$$\sin(ax^\circ + b^\circ), \qquad \frac{1}{(1+x)\sqrt{(1+x^2)}}.$$

(*b*) Evaluate

$$\int_0^a x\sqrt{\left(\frac{a-x}{a+x}\right)}\,dx, \qquad \int_1^e \log_e x\,dx.$$

9. By integration by parts or otherwise, prove that, except for a constant of integration,

$$(n-1)\int \frac{\sin\theta\,d\theta}{(a\sin\theta+b\cos\theta)^n}$$
$$= \frac{\cos\theta}{(a\sin\theta+b\cos\theta)^n} + na\int \frac{d\theta}{(a\sin\theta+b\cos\theta)^{n+1}}.$$

Putting $\quad I_n = \displaystyle\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(a\sin\theta+b\cos\theta)^n},$

$$S_n = \int_0^{\frac{1}{2}\pi} \frac{\sin\theta\,d\theta}{(a\sin\theta+b\cos\theta)^n},$$

$$C_n = \int_0^{\frac{1}{2}\pi} \frac{\cos\theta\,d\theta}{(a\sin\theta+b\cos\theta)^n},$$

prove that

$$(n-1)S_n = naI_{n+1} - \frac{1}{b^n} \quad \text{and} \quad (n-1)C_n = nbI_{n+1} - \frac{1}{a^n}.$$

By considering $aS_n + bC_n$, or otherwise, find a recurrence relation between I_{n+1} and I_{n-1}.

10. Sketch the curve $16y^2 = x^2(2-x^2)$, explaining your reasons for the main features that you indicate.

Show that the coordinates of a point on this curve may be expressed in terms of a parameter θ by the equations

$$x = \sqrt{2}\sin\theta, \qquad y = \tfrac{1}{2}\sin\theta\cos\theta.$$

Calculate the total arc length of the curve and the area of the surface of revolution generated by the rotation of one of its loops through two right angles about the x-axis.