A 51—II

OXFORD LOCAL EXAMINATIONS

General Certificate of Education

Summer Examination, 1964

Advanced Level

PURE MATHEMATICS, PAPER II

Thursday, 25 June. Time allowed: 3 hours

Write the number of the paper, **A 51/II**, *on the left at the head of each sheet of your answers in the space provided.*

Complete answers to a few questions receive more credit than scrappy attempts at the whole paper. A pass may be obtained on good answers to about **four** *questions.*

Mathematical tables are provided.

1. Prove that, if $t = \tan\frac{1}{2}\theta$,

$$\sin\theta = \frac{2t}{1+t^2} \quad \text{and} \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$

If β, γ are solutions of the equation

$$p\cos\theta + q\sin\theta = r$$

which do not differ by an integral multiple of 2π, prove that

$$\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma = \frac{r-p}{r+p} \quad \text{and} \quad \tan\tfrac{1}{2}\beta + \tan\tfrac{1}{2}\gamma = \frac{2q}{r+p}.$$

If β, γ are two such solutions of the equation for θ

$$a^2\cos\theta + a\sin\theta + 1 = 0,$$

prove that

$$a^2\cos\beta\cos\gamma + a(\sin\beta + \sin\gamma) + 1 = 0.$$

2. The bisectors of the angles B and C of triangle ABC meet the opposite sides at E and F. Find the projections of EF on BC and perpendicular to BC and prove that the acute angle between EF and BC is

$$\tan^{-1}\frac{|b-c|\sin A}{(a+b)\cos C+(a+c)\cos B}.$$

3. Two vertical towers of heights h ft. and k ft. stand on level ground. To a man at a point D on the line joining the feet B and C of the towers, the angles of elevation of the tops of the towers are both α. On walking a ft. along a straight horizontal path inclined at an acute angle β to the line joining the feet of the towers, to a point A, the man finds that the angles of elevation of the tops of the towers are both γ. Prove that AD bisects angle BAC and that

$$\left|\frac{a}{h}-\frac{a}{k}\right| = 2\cot\alpha\cos\beta \quad\text{and}\quad \frac{a^2}{hk} = \cot^2\gamma-\cot^2\alpha.$$

4. The perpendiculars from the vertices of triangle ABC to the opposite sides meet at P. Prove that angle BPC is $180^\circ-A$.

Prove also that the circumcircles of triangles BPC, CPA, APB have equal radii and that their centres form a triangle congruent to ABC and having P as its circumcentre.

5. Spheres are drawn each passing through two fixed points, P and Q, and touching a fixed plane p. P and Q are on the same side of p. Prove that the locus of the point of contact of the spheres with the plane is a circle whose centre is the point of intersection of PQ with p.

Prove also that the centres of the spheres lie on the curve of intersection of a certain right circular cylinder, whose axis is perpendicular to p, with a certain plane perpendicular to PQ.

6. Prove that the coordinates of any point on the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\alpha, b\sin\alpha)$ can be put in the form

$$((1-b^2t)a\cos\alpha,\ (1-a^2t)b\sin\alpha),$$

and show that the distance between the points for which the values of t are t_1 and t_2 is

$$ab(t_1-t_2)\sqrt{(b^2\cos^2\alpha+a^2\sin^2\alpha)}.$$

If the normal meets the ellipse again at R, prove that

$$PR = \frac{2ab(b^2\cos^2\alpha+a^2\sin^2\alpha)^{\frac{3}{2}}}{b^4\cos^2\alpha+a^4\sin^2\alpha}.$$

7. Find the equation of the chord of the parabola $y^2 = 4ax$ joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, and show that it intersects the axis of the parabola at the point $(-at_1t_2, 0)$.

XY is any chord of a parabola through the point $(5a, 0)$ The circle on XY as diameter cuts the parabola again at Z and W. Prove that ZW passes through the focus of the parabola.

8. The parabola $y^2 = 4ax$ and the hyperbola $4xy = a^2$ meet at P. Their common tangent touches the parabola at Q and the hyperbola at R. Prove that PR is the tangent at P to the parabola and that QP is the tangent at P to the hyperbola.

9. Find the equation of the tangent to the curve

$$y = a\sin^3\theta, \qquad x = a\cos^3\theta$$

at the point $\theta = \alpha$. Show that the tangents at the points α, $\alpha+\frac{1}{2}\pi$, $\alpha+\pi$ and $\alpha+\frac{3}{2}\pi$ form a square whose diagonals are $y+x\tan(\alpha+\frac{1}{4}\pi) = 0$ and $y+x\tan(\alpha+\frac{3}{4}\pi) = 0$.

10. Show that the equation in polar coordinates, $r = a\cos\theta$, represents a circle, and find where it meets the curve

$$r = a(1+\cos 2\theta).$$

Trace the curves and calculate the area common to the circle and a loop of the other curve.